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REPRESENTATIONS OF FUNDAMENTAL GROUPS WHOSE HIGGS BUNDLES ARE PULLBACKS

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Abstract

The representations of the fundamental group of a smooth projective variety into a complex simple group are discussed in terms of the corresponding Higgs bundles. A necessary and sufficient condition is found for a representation to factor geometrically through the fundamental group of an orbicurve. The factorization question is studied further for the case of higher dimensional target varieties.

1. Introduction

The relationship between the representations of the fundamental group of a smooth projective variety and its Higgs bundles is well understood by now. In recent years the efforts of many remarkable mathematicians to put this relationship into action resulted in a new approach to the theory of the moduli spaces and lead to a major breakthrough in the long-lasting attempts for developing a nonabelian Hodge theory. The initial step belongs to Hitchin, who established this relationship in the case of algebraic curves and for representations in $SL(2, \mathbb{C})$ (see [15]). After that the relation has been studied in detail by Simpson [21], [22], [23] and Corlette [6], [8] in higher dimensions and for representations in arbitrary simple Lie group. In his work [10] Donagi developed the theory of the Hitchin maps in higher dimension and gave complete geometric description of their fibers.

Recently Simpson [23] has proven that every nonrigid Zariski dense representation of the fundamental group of a smooth projective variety in $SL(2, \mathbb{C})$ factors through the fundamental group of some orbicurve. This theorem is an analogue of a theorem of Culler and Shalen [9] and is motivated by the works of Gromov [13], Yau and Jost [17], Green and Lazarsfeld [12], Carlson and Toledo [4], and Goldman and Millson [11]. The present paper contains our attempts to investigate the same question for other simple Lie groups. Using Simpson's ideas and Donagi's theory

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we will show some necessary and sufficient conditions for such a representation to factor through the fundamental group of an orbicurve as well as some analogous results for the representations which factor through the representations of the fundamental group of higher dimensional varieties.

2. Moduli spaces of Higgs bundles and representations: basic facts

2.1. Let S be a smooth polarized projective variety over $\mathbb{C}$. Let G be a simple complex Lie group with a Lie algebra $\mathfrak{g}$. A Higgs bundle on S is a pair (E, θ) consisting of a holomorphic principal G bundle E and a $\mathfrak{g}_E$-valued 1-form $\theta \in H^0(S, \mathfrak{g}_E \otimes \Omega^1_S)$ such that $\theta \wedge \theta = 0$, where $\mathfrak{g}_E$ is the bundle of Lie algebras associated to E, i.e., $\mathfrak{g}_E = E \times_{\mathrm{Ad}} \mathfrak{g}$. A Higgs bundle is stable if for any subsheaf $U \subset \mathfrak{g}_E$, preserved by θ, the standard inequality $\mu(U) < \mu(\mathfrak{g}_E)$ for the slopes of U and $\mathfrak{g}_E$ holds. (Here the slopes $\mu(U)$ and $\mu(\mathfrak{g}_E)$ are computed with respect to the fixed polarization on S.) The nonabelian version of the Hodge theorem [20] establishes a one-to-one correspondence between the set of stable Higgs bundles with vanishing Chern classes and structure group G and the set of all irreducible representations of the fundamental group $\pi_1(S)$ of S in G. In [20] Simpson constructed the moduli space of Higgs bundles $\mathscr{M}_{\mathrm{Higgs}}$ which (cf. [23, Theorem 2]) is homeomorphic to $\mathscr{M}_{\mathrm{Rep}}$—the moduli space of the irreducible representations of $\pi_1(S)$ in a simple Lie group—an affine variety, constructed for example in [9].

There is another moduli space on S related to $\mathscr{M}_{\mathrm{Higgs}}$—the space $\mathscr{U}_G$ of all stable principal G-bundles over S. In general, $\mathscr{M}_{\mathrm{Higgs}}$ has many smooth irreducible components (cf. [25]) which are holomorphic symplectic manifolds. However, when S is a compact curve Simpson [22] has proven that $\mathscr{M}_{\mathrm{Higgs}}$ is integral. In this particular case Hitchin has shown that $\mathscr{M}_{\mathrm{Higgs}}$ is a partial compactification of the cotangent bundle of $\mathscr{U}_G$ (for more details see [15], [3]). In his recent survey [7] Corlette observed that for $\dim S > 1$ the same property holds for an irreducible component of $\mathscr{M}_{\mathrm{Higgs}}$ of maximal dimension.

2.2. Let $\rho\colon G \to \mathrm{End}(V)$ be any irreducible representation. Using ρ one can associate to each Higgs pair (E, θ) a vector bundle $\mathbf{V} = E \times_\rho V$ and a twisted endomorphism $\rho(\theta)\colon \mathbf{V} \to \mathbf{V} \otimes \Omega^1_S$. The condition $\theta \wedge \theta = 0$ allows us to interpret any G-invariant homogeneous polynomial $f \in \mathbb{C}[\mathrm{End}(V)]^G$ of degree d as a holomorphic map

$$f\colon \mathscr{M}_{\mathrm{Higgs}} \to H^0(S, \mathrm{Symm}^d \Omega^1_S), \qquad (E, \theta) \to f(\rho(\theta)).$$

Definition 2.1. Let $f_1, \cdots, f_m$ be a base of the ring of invariants $\mathbb{C}[\mathrm{End}(V)]^G$ consisting of homogeneous polynomials of degrees $d_1, \cdots, d_m$ respectively. The holomorphic map

$$\mathcal{H}_\rho : \mathcal{M}_{\mathrm{Higgs}} \to \bigoplus_{i=1}^{m} H^0(S, \mathrm{Symm}^{d_i}\Omega_S^1)$$

is called the Hitchin map associated to the representation ρ.

Investigation of the structure of the Hitchin maps turns out to be extremely helpful for a better understanding of the geometry of the moduli space of all Higgs bundles. For the case where S is a curve, they were introduced and studied by Hitchin (see [14], [16]) who proved that for the standard representations of the classical Lie groups their fibers are Jacobians or Prym varieties of certain coverings of S. Moreover his analysis showed that the fibers of $\mathcal{H}$ are Lagrangian submanifolds in the moduli space of Higgs bundles, i.e., it is an algebraically completely integrable system. Again for a curve but for any semisimple Lie group in the case where ρ is a regular representation, the fibers of the Hitchin map were studied by Beilson and Kazhdan [3]. For a higher dimensional S, Simpson [21] proved that $\mathcal{H}_\rho$ is a proper holomorphic map when ρ is the standard representation of $SL(k, \mathbb{C})$. Finally, Donagi [10] described the fibers of the Hitchin maps for any dimension of S and for any representation. They are projective varieties (typically, abelian varieties) associated to coverings of S, the so-called spectral coverings.

2.3. There is a natural $\mathbb{C}^*$-action on $\mathcal{M}_{\mathrm{Higgs}} t: (E, \theta) \to (E, t\theta)$. The locus of all fixed points of this action consists exactly of so-called complex variations of the Hodge structure (cf. [20]) which are studied in detail in a series of works of Simpson [20], [21], [22]. One of its most interesting features is that the corresponding locus in $\mathcal{M}_{\mathrm{Rep}}$ contains all rigid representations of $\pi_1(S)$ in G, that is the representations for which every nearby representation is conjugate to them.

In the next sections we will characterize those nonrigid representations of the fundamental group of S which are of geometric origin, namely which factor through the fundamental group of some orbicurve Y.

3. Higgs fields and spectral covers: an outline of Donagi's theory

In this section we will describe shortly some recent results of Donagi concerning the general theory of spectral coverings. Since this theory is fairly extensive we will concentrate only on the parts we will need afterwards.

3.1. Let (E, θ) be a principal irreducible Higgs bundle over S. The holomorphic section $\theta \in H^0(S, \mathfrak{g}_E \otimes \Omega_S^1)$ gives a homomorphism of vector bundles $\mathrm{ad}_\theta : \mathfrak{g}_E \to \mathfrak{g}_E \otimes \Omega_S^1$.

Definition 3.1. The Higgs bundle (E, θ) is said to be *regular* when the coherent subsheaf $\ker(\mathrm{ad}_\theta) \subset \mathfrak{g}_E$ has rank equal to the rank of the Lie algebra $\mathfrak{g}_E$. Similarly (E, θ) is said to be *regular and semisimple* if there exists a Zariski open subset $S_0 \subset S$ for which $\ker(\mathrm{ad}_\theta)|_{S_0}$ is a subbundle of Cartan subalgebras in $\mathfrak{g}_E|_{S_0}$.

Choosing a local trivialization for E and a system $z_1, \cdots, z_n$ of local parameters on S, one can always write θ in the form $\theta = \theta_1\, dz_1 + \cdots + \theta_n\, dz_n$. The condition $\theta \wedge \theta = 0$ yields $[\theta_i, \theta_j] = 0$ for each i and j. Hence the regularity in the above definition simply means that for a generic point x the annulator of the linear span of the elements $\theta_i(x)$ is of the minimal possible dimension. Furthermore, θ is regular and semisimple if this annulator is a Cartan subalgebra. In the same way one can define semisimple (respectively nilpotent) Higgs bundle requiring all θ_i to be semisimple (respectively nilpotent) at a generic point.

Remark 3.1. One can easily construct a Higgs bundle (actually direct product of Higgs $\mathbb{C}^*$-bundles) which is regular and semisimple. Since the property of an element to be regular and semisimple is an open condition this yields that when S is a curve the subset of all regular semisimple Higgs bundles in $\mathcal{M}_{\mathrm{Higgs}}$ is Zariski open and dense. When the dimension of S is higher, the moduli space is reducible, and it is unclear whether each of its components contains a regular and semisimple element.

With each (everywhere) regular Higgs bundle (E, θ) one can associate a finite Galois covering $\widetilde{S} \to S$ which is constructed abstractly and plays the role of an archetypical model for all spectral coverings in Donagi's theory. For simplicity we describe the construction of $\widetilde{S}$ only for regular and semisimple Higgs bundles which are everywhere regular (for more details and for the treatment of the regular case see [10]). If (E, θ) is such a bundle, we denote the subsheaf $\ker(\mathrm{ad}_\theta) \subset \mathfrak{g}_E$ by $\mathfrak{t}_E$. The fact that (E, θ) is everywhere regular implies that $\mathfrak{t}_E$ is locally free. By definition there exists a Zariski open set S_0 in S over which $\mathfrak{t}_E$ is a subbundle of Cartan subalgebras in $\mathfrak{g}_E$. Similarly if $G_E := E \times_{\mathrm{Ad}} G$ is the bundle of groups associated with E, then denote by $\mathfrak{T}_E \subset G_E$ the group subscheme stabilizing the section $\theta \in H^0(S, \mathfrak{g}_E \otimes \Omega_S^1)$ under the natural action of G_E on $\mathfrak{g}_E$.

The principal bundle E is furnished with two natural actions:

(i) The right action of the trivial bundle of groups G (with respect to which E is a principal homogeneous G space): $E \times G \to E$;

(ii) The left action of the bundle G_E (G_E can be viewed as the bundle of holomorphic gauge transformations of E): $G_E \times E \to E$.

Fix a maximal torus $\mathfrak{T} \subset G$. Consider the subbundle $\mathfrak{N}_E \subset E$ given by

$$(\mathfrak{N}_E)_s = \{e \in E_s | (\mathfrak{T}_E)_s e = e\mathfrak{T}\},$$

for every $s \in S$. It is easy to see that $\mathfrak{N}_E$ is a principal $N(\mathfrak{T})$ bundle over S_0. Let $\widetilde{S}$ be the Stein factorization of the map $\mathfrak{N}_E \to S$. Obviously $\widetilde{S}$ is a Galois cover of S with Galois group W, the Weyl group of the Lie algebra $\mathfrak{g}$.

Definition 3.2. The cover $\widetilde{S}$ is called *a spectral cover associated with the pair* (E, θ).

The covering $\widetilde{S}$ may be reducible despite the fact that (E, θ) is irreducible. The set of all irreducible components of $\widetilde{S}$ is permuted by a subgroup of W which measures how far is the element θ from the semisimplicity. When (E, θ) is regular and semisimple, the spectral cover is irreducible.

All elements in one and the same fiber of any Hitchin map have birationally equivalent abstract spectral coverings. The spectral cover $\widetilde{S}$ in general is not smooth even for regular and semisimple elements. The singularities of $\widetilde{S}$ contain essential information about the geometry of $\mathcal{M}_{\text{Higgs}}$ and cannot be disregarded. However, for the generic fiber consisting only of regular and semisimple Higgs bundles which are everywhere regular, one can show that $\widetilde{S}$ is smooth and is the same for all bundles in the fiber.

Remark 3.2. The modification of all of the above considerations for reducible Higgs bundles is straightforward. Another, subtler part of Donagi's theory deals with the case of the nonregular bundles which we will not need here.

3.2. The archetype $\widetilde{S}$ mentioned above has different incarnations corresponding to a choice of representation ρ of G and hence to a choice of a Hitchin map. For each irreducible ρ one can define a map over S_0:

$$\begin{array}{ccc} \widetilde{S}|_{S_0} & \xrightarrow{\pi_\rho} & \mathfrak{t}_E^\vee|_{S_0} \\ & \searrow \quad \swarrow & \\ & S_0 & \end{array}$$

which depends on ρ. The Zariski closure S_ρ of the image $\pi_\rho(\widetilde{S}|_{S_0})$ is an integral subscheme in the total space of the sheaf $\mathfrak{t}_E$, the spectral covering associated with the data $((E, \theta), \rho)$. This covering has an independent definition which we proceed to describe. Having ρ one produces a multisection $\mathfrak{s}_\rho$ in the bundle $\mathfrak{t}_E^\vee|_{S_0}$ by assigning to each point $s \in S_0$ the set $\mathfrak{s}_\rho(s) \subset (\mathfrak{t}_E^\vee)_s$ of all extremal weights for the representation ρ in the dual to the Cartan algebra $(\mathfrak{t}_E)_s$. The image $\mathfrak{s}(S_0)$ is a covering of S which is exactly $S_\rho|_{S_0}$.

In the case of representation with a regular highest weight, one can prove that the map π_ρ is a birational isomorphism between $\widetilde{S}$ and $\widetilde{S}_\rho$. In general, the covering $\widetilde{S} \xrightarrow{\pi_\rho} S_\rho$ is Galois covering with a group isomorphic to the stabilizer of the highest weight of ρ.

The main theorem of Donagi [10] describes the generic fiber of $\mathscr{H}_\rho$ consisting of regular and semisimple Higgs bundles in terms of the spectral covering S_ρ. Each connected component of such a fiber can be identified with a certain abelian subvariety $\mathrm{Prym}_\rho(S_\rho, S)$ in $\mathrm{Pic}^0(S_\rho)$, a generalized Prym variety. For representations with regular highest weight we can think about $\widetilde{S}$ as a smoothing of S_ρ. In this case $\mathrm{Prym}(\widetilde{S}, S)$ is essentially the connected component of the identity of the commutative algebraic group $\mathrm{Hom}_{\mathbb{Z}[W]}(\chi(\mathscr{T}), \mathrm{Pic}^0(\widetilde{S}))$, where $\chi(\mathscr{T})$ is the character lattice for some maximal torus $\mathscr{T} \subset G$ considered as a W module. It can be realized (not canonically) as an abelian subvariety in $\mathrm{Pic}^0(\widetilde{S})$ by picking up a primitive element χ_0 in the irreducible W module $\chi(H)$ and then assigning to each homomorphism from $\mathrm{Hom}_{\mathbb{Z}[W]}(\chi(H), \mathrm{Pic}^0(\widetilde{S}))$ its value at χ_0. These varieties are analogues of the classical Prym varieties for two-sheeted coverings of curves [1] or to the Prym-Turin varieties which appear in the theory of threefolds [24] and play an important role in the linearlization problems for the algebraically completely integrable systems [18].

In his work [10] Donagi compares the connected components of the fibers of different $\mathscr{H}_\rho$ passing through a fixed point $(E, \theta) \in \mathscr{M}_{\mathrm{Higgs}}$. This comparison clarifies the behavior of (E, θ) when represented as a Higgs vector bundle via some ρ. These different Prym varieties turn out to be isogeneous, and their isogeny is of geometric nature, coming from correspondences between the different incarnations of $\widetilde{S}$.

When the fiber of the Hitchin map is regular and semisimple but not generic and has more than one (singular) abstract spectral covering, one can stratify the fiber, so that each strata will be Prym for one of the coverings (probably noncompact), and the biggest strata corresponds to the covering with worse singularities and the smallest one to the smooth model. (This phenomenon was observed by Hitchin for $SL(2, \mathbb{C})$ Higgs bundles over a curve [14].) In the regular and nonsemisimple fibers and in the nonregular fibers the picture is much more complicated.

Remark 3.3. One can define generalized Prym variety (mimicking the one considered above) starting from any free irreducible W module Λ over $\mathbb{Z}$. Indeed set $\mathrm{Prym}_{\Lambda}(\widetilde{S}, S)$ to be the identity component of the group $\mathrm{Hom}_{\mathbb{Z}[W]}(\Lambda, \mathrm{Pic}^0(\widetilde{S}))$. Again picking up a primitive element for Λ we can imbed this Prym variety in to $\mathrm{Pic}^0(\widetilde{S})$. The analysis of the geometric structure of the Picard variety of $\widetilde{S}$ done by Donagi (see [10]) shows that $\mathrm{Pic}^0(\widetilde{S})$ is actually isogenous to a direct sum of Prym varieties of this type (with multiplicities) for modules Λ which are submodules in the group ring $\mathbb{Z}[W]$.

An interesting question posed by Donagi is to find the relation of these other pieces of $\mathrm{Pic}^0(\widetilde{S})$ to the moduli space $\mathcal{M}_{\mathrm{Higgs}}$. As we shall see later, the problem of characterizing the representations which factor through some curve can be expressed as the condition about nontriviality of one of these Prym varieties.

3.3. There is another convenient realization of the spectral coverings. The Higgs bundle (E, θ) provides us with one more natural map:

$$i_\theta : \mathfrak{t}_E^{\vee} \to \Omega_S^1, \qquad \varphi \to \varphi(\theta).$$

The image $i_\theta(S_\rho)$ is again a covering of S which lies in the total space of the vector bundle Ω_S^1. For regular and semisimple Higgs bundles the map $i_\theta : S_\rho \to i_\theta(S_\rho)$ is clearly a birational isomorphism since in this case the operator $\rho(\theta)(x)$ will be regular and semisimple for generic $x \in S$, i.e., all of its different extremal eigenvalues will have multiplicity one.

We can look at the covering $i_\theta(S_\rho)$ from a different angle. Consider the vector Higgs bundle $(\mathbf{V}, \rho(\theta))$ associated to (E, θ) by the representation ρ. Let $\mathrm{Tot}(\Omega_S^1)$ be the total space of Ω_S^1, and $\lambda \in H^0(\mathrm{Tot}(\Omega_S^1), p^*(\Omega_S^1))$ be the tautological section. Because of the condition $\theta \wedge \theta = 0$ we can consider $\rho(\theta)$ as an endomorphism of $\mathbf{V}$ with coefficients at the symmetric algebra of Ω_S^1, and form the element $\det(\rho(\theta) - \lambda \cdot \mathrm{id}_{\mathbf{V}}) \in H^0(S, \mathrm{symm}^* \Omega_S^1)$. The zero scheme of the section $\det(\rho(\theta) - \lambda \cdot \mathrm{id}_{\mathbf{V}})$ is a subscheme in $\mathrm{Tot}(\Omega_S^1)$ which is always finite over S. It is in general reducible, and the irreducible components correspond to the splitting of the set of all weights for ρ as a union of orbits of W_G (cf. [10]). The irreducible component corresponding to the class of all extremal weights coincides with the covering $i_\theta(S_\rho)$.

Observe that when the highest weight for ρ is minuscule, it has only extremal part and therefore the zero scheme of the section

$\det(\rho(\theta) - \lambda \cdot \mathrm{id}_V)$ is irreducible. For instance this is the case with the standard representations of the classical series A_n, D_n or with the exceptional series E_6, E_7, E_8. The spectral coverings for such representations were considered in [14], [2], [18] in the case of curves and in [21] in higher dimensions.

When the representation ρ is with regular highest weight, the spectral covers S_ρ and $i_\theta(S_\rho)$ are birationally isomorphic to $\widetilde{S}$ and are Galois with group W. Abusing the notation we will denote the smoothing of $i_\theta(S_\rho)$ by $\widetilde{S}$. This covering will play a central role in our subsequent considerations.

4. Constructions and theorems

4.1. To make the exposition clearer we assume throughout this section that the regular and semisemiple Higgs bundle (E, θ) is everywhere regular and therefore has smooth Galois spectral cover $p: \widetilde{S} \to S$. This restriction is nonessential and can be removed (see Remark 4.4).

Let $\Lambda \subset \mathbb{Z}[W]$ be an irreducible W-submodule such that the linear representation $\Lambda \otimes_{\mathbb{Z}} \mathbb{C}$ is also irreducible. Consider the generalized Prym variety $\mathrm{Prym}_\Lambda(\widetilde{S}, S)$. If we pick up a primitive element $\xi \in \Lambda$ and imbed $\mathrm{Prym}_\Lambda(\widetilde{S}, S)$ in $\mathrm{Pic}^0(\widetilde{S})$ by this element, it is clear that the tangent space at the point $0 \in \mathrm{Pic}^0(\widetilde{S})$ to the imbedded variety will be the image of the linear map

$$\mathrm{Hom}_{\mathbb{C}[W]}(\Lambda \otimes_{\mathbb{Z}} \mathbb{C},\ T_0\mathrm{Pic}^0(\widetilde{S})) \to T_0\mathrm{Pic}^0(\widetilde{S}), \qquad f \to f(\xi).$$

Therefore if the module $\Lambda \otimes_{\mathbb{Z}} \mathbb{C}$ participates in $T_0\mathrm{Pic}^0(\widetilde{S})$ with multiplicity k, and the dimension $\dim_{\mathbb{C}}(\Lambda \otimes_{\mathbb{Z}} \mathbb{C}) = m$, then we can conclude by Schur's lemma that $\dim(\mathrm{Prym}_\Lambda(\widetilde{S}, S)) = k$ and that this particular Prym variety participates in the isogeneous decomposition of $\mathrm{Pic}^0(\widetilde{S})$ with multiplicity m.

Example 4.1. If $\Lambda = \chi(H)$ is the character lattice for some maximal torus H in G, then $\Lambda \otimes_{\mathbb{Z}} \mathbb{C} = \mathfrak{t}^\vee = \mathfrak{t}$ as a W module. (The last equality holds since the representation $\mathfrak{t}$ of W is orthogonal.) Thus the Prym variety describing the connected components of the fiber of a Hitchin map always has multiplicity equal to the rank of the Lie algebra $\mathfrak{g}$ in the isogenous decomposition of $\mathrm{Pic}^0(\widetilde{S})$.

Example 4.2. If Λ is a rank-one representation of W, the corresponding prymian will have multiplicity one. In this case $\mathrm{Prym}_\Lambda(\widetilde{S}, S)$ obviously has unique imbedding as a subvariety in $\mathrm{Pic}^0(\widetilde{S})$, and the corresponding image is a W invariant abelian subvariety. For example, when $\Lambda = \mathbb{Z}$ is the trivial W module, the imbedded prymian consists of all line bundles of degree zero on $\widetilde{S}$ which are invariant under the action of W and hence coincides with $p^*\mathrm{Pic}^0(S)$.

Another interesting instance is when Λ is the sign representation of W. The Weyl group W is generated by reflections, and we have the so-called length function l which assigns to each element $w \in W$ the length $l(w)$ of a minimal decomposition of w as a product of reflections. The sign representation ε is isomorphic to $\mathbb{Z}$ with the W-action given by $w \to (n \mapsto (-1)^{l(w)} n)$. The corresponding Prym variety $\mathrm{Prym}_\varepsilon(\widetilde{S}, S)$ can be described as the intersection of all classical Prym varieties associated with the two sheeted coverings $\widetilde{S} \xrightarrow{\sigma} \widetilde{S}/\sigma$, σ being a reflection in W.

4.2. Our goal in this section is to characterize the representations of $\pi_1(S)$ which can be factored geometrically through some curve. First we will need the following definition.

Definition 4.1. An *orbicurve* is a triple $(Y, \{x_i\}, \{n_i\})$ consisting of a smooth projective curve Y, a finite set of points $\{x_i\} \subset Y$, and a set of positive integers $\{n_i\}$ prescribed to the points. The fundamental group of the orbicurve $(Y, \{x_i\}, \{n_i\})$ is by definition the fundamental group of the noncompact curve $Y \backslash \{x_i\}$ factorized by the additional relations requiring the order of the simple loops around the points x_i to be exactly n_i.

Each orbicurve is "uniformized" by the Poincaré disk Δ^1 in the following sense. There exists a unique (up to isomorphism) infinite Galois covering $\Delta^1 \to Y$ ramified exactly over the points x_i with ramification degree n_i respectively. The Galois group of this covering is a Fuchsian group Γ generated by the $2g$ hyperbolic elements which generate the fundamental group of Y plus some elliptic elements γ_i such that γ_i generates the stabilizer of some ramification point over x_i. Each γ_i is obviously of order n_i, and the fundamental group of the orbicurve is isomorphic to Γ.

Definition 4.2. A *representation* $\varrho : \pi_1(S) \to G$ factors geometrically through some orbicurve $(Y, \{x_i\}, \{n_i\})$ when there exists a regular map $f : S \to Y$ inducing a homomorphism $f_* : \pi_1(S) \to \Gamma$ and a representation ϱ' of Γ in G such that $\varrho' \circ f_* = \varrho$.

In terms of Higgs bundles this means just that the pair (E, θ) which corresponds to ϱ is a pullback via f of some principal Higgs bundle (E_Y, θ_Y) over the open curve $Y \backslash \{x_i\}$. (The number $\{n_i\}$ prescribe the holonomy for the Hodge connection in (E_Y, θ_Y).)

The following theorem gives a necessary and sufficient condition for a representation to factor geometrically.

Theorem 4.1. *Let* ϱ *be a Zariski dense representation of* $\pi_1(S)$ *whose Higgs bundle* (E, θ) *is regular and semisimple. The* ϱ *factors geometrically through an orbicurve if and only if* $\mathrm{Prym}_\varepsilon(\widetilde{S}, S) \neq 0$, *where* $\widetilde{S}$ *is the smooth spectral cover for* (E, θ).

Proof. 1. To prove the "if" part we need to construct a map $f: S \to Y$ to some (preferably nonrational) curve. A standard way of finding such a map is to modify the Albanese map $A: S \to \mathrm{Alb}(S)$ for S. This approach, however, has one disadvantage at its very beginning: one should be able to prove a priori that $\mathrm{Alb}(S)$ is nontrivial, i.e., that S has nontrivial holomorphic 1-form. Since we do not have any specific geometric information about S, such a verification is difficult. To circumvent this difficulty Simpson (see [23]) suggests to look for a map from the spectral covering $\widetilde{S}$ to a suitable curve.

Observe first that on $\widetilde{S}$ there exists a natural nontrivial holomorphic 1-form ω which is just the restriction (and projection) of the tautological 1-form λ on $\mathrm{Tot}(\Omega^1_S)$. (The nontriviality of ω follows form the definition of $\widetilde{S}$; compare with §3.3.) Thus the nontriviality of $\mathrm{Alb}(\widetilde{S})$ is guaranteed and we can use the Albanese map for $\widetilde{S}$.

Let A be a W-equivariant Albanese map for $\widetilde{S}$. One can always choose such a map requiring the initial point for A in the Chow group of 0-cycles on $\widetilde{S}$ to be a W-equivariant cycle—for instance, the sum of all points in some nonramified fibre of the covering $\widetilde{S} \to S$. The image of $\widetilde{S}$ under A will be a W-invariant subvariety of the Albanese variety $\mathrm{Alb}(\widetilde{S})$. The dimension of this image will be in general quite big, so we need to modify further the Albanese map.

By choosing a W-invariant polarization on $\widetilde{S}$ we can view $T_0(\mathrm{Pic}^0(\widetilde{S}))$ and $T_0(\mathrm{Alb}(\widetilde{S}))$ as dual W modules. Since the sign representation ε is self-dual and by hypothesis $\mathrm{Prym}_\varepsilon(\widetilde{S}, S) \neq 0$, we obtain that ε participates in the irreducible decomposition of $T_0(\mathrm{Alb}(\widetilde{S}))$ with a positive multiplicity. Hence the abelian variety $\mathscr{P} = \mathrm{Hom}_{\mathbb{Z}[W]}(\varepsilon, \mathrm{Alb}(\widetilde{S}))$ can be imbedded uniquely as a W-invariant subvariety in $\mathrm{Alb}(\widetilde{S})$ of positive dimension. Again one can think of it as the intersection of the antiinvariant parts of the Albanese variety with respect to the reflections in W.

There is a natural projection $\mathrm{Alb}(\widetilde{S}) \xrightarrow{P_\varepsilon} \mathscr{P}$ defined by

$$P_\varepsilon = \sum_{w \in W} (-1)^{l(w)} w.$$

Set $P: \mathrm{Alb}(\widetilde{S}) \to \mathscr{P}/W$ to be the composition of P_ε with the factorization map $\mathscr{P} \to \mathscr{P}/W$.

Let D be the ramification divisor of the spectral covering $\widetilde{S}$, and $P(A(D))$ be its image in $\mathscr{P}/W$.

Lemma 4.1. 1. $P(A(\mathrm{supp}(D))) = \{0\} \subset \mathscr{P}/W$.

2. *There exists a holomorphic map* $\varphi: S \to \mathscr{P}/W$ *which makes the following diagram commutative:*

$$\begin{array}{ccc} \widetilde{S} & \xrightarrow{A} & \mathrm{Alb}(\widetilde{S}) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle P} \\ S & \xrightarrow{\varphi} & \mathscr{P}/W \end{array}$$

Proof. The proof of the first statement is based on the fact that every point of the ramification divisor is fixed by some reflection. Indeed after fixing a Cartan subalgebra $\mathfrak{t}$ for each $x \in \mathrm{supp}(D)$ we can identify the nontrivial stabilizer $\mathrm{St}_W(x)$ with the stabilizer of some (nonregular) weight $\psi \in \mathfrak{t}^\vee_{\mathbb{R}}$. There is a unique face F of a Weyl chamber in $\mathfrak{t}^\vee_{\mathbb{R}}$ of minimal possible dimension, which contains ψ in its interior. The face F is intersection of hyperplanes corresponding to simple reflections in W, and the stabilizer $\mathrm{St}_W(\psi)$ is generated by those reflections. Hence $\mathrm{St}_W(x) = \mathrm{St}_W(\psi)$ contains a reflection.

Let now $\sigma \in W$ be an arbitrary reflection. The projector P_ε decomposes in the group ring $\mathbb{Z}[W]$ as a product:

$$P_\varepsilon = \left(\sum_{w \in W_{\mathrm{even}}} w \right) \cdot (\mathrm{id} - \sigma).$$

This implies that $P_\varepsilon(A(xj)) = 0$ for any point $x \in \widetilde{S}$ invariant under σ. Combined with the previous observation this yields $P(A(\mathrm{supp}(D))) = \{0\}$.

The second statement of the lemma holds because by construction $P_\varepsilon \circ A$ is a W-equivariant map. q.e.d.

The holomorphic map φ obtained above is a natural candidate for a factorizing morphism. To study the dimension of its image observe first that all holomorphic 1-forms of type ε on $\widetilde{S}$ by construction come as pullbacks of holomorphic 1-forms under the map $P_\varepsilon \circ A: \widetilde{S} \to \mathscr{P}$. (Recall that $T_0\mathscr{P} \subset T_0\mathrm{Alb}(\widetilde{S}) = H^0(\widetilde{S}, \Omega^1_{\widetilde{S}})$ consists exactly of all forms of type ε.) Since by hypothesis $\dim \mathscr{P} \geq 1$ we get $\dim P_\varepsilon \circ A(\widetilde{S}) \geq 1$. But the image $\varphi(S)$ is just a quotient of $(P_\varepsilon \circ A)(\widetilde{S})$ under the action of the finite group W and hence has dimension strictly greater than zero.

Lemma 4.2. *The image $\varphi(S)$ is one dimensional.*

Proof. Suppose that $d = \dim \varphi(S) > 1$ and let $\dim S = n$. By Noether's normalization theorem we can find a generically finite map $\Phi: S \to \mathbb{P}^{n-d} \times \varphi(S)$ which makes the following diagram commutative:

$$\begin{array}{ccc} S & \xrightarrow{\Phi} & \mathbb{P}^{n-d} \times \varphi(S) \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle p_2} & \\ & \varphi(S) & \end{array}$$

Let

$$\begin{array}{ccc} & N & \\ {\scriptstyle \text{birational}}\nearrow & & \searrow{\scriptstyle \text{finite}} \\ S & & \mathbb{P}^{n-d} \times \varphi(S) \end{array}$$

be the Stein factorization of the map Φ. Denote by F the birational morphism $S \to N$. Since N is normal, the rational map $F^{-1}: N \to S$ is not defined on a locus $A \subset N$ of codimension 2. By Lemma 4.1 the branch locus $R \subset S$ of the covering $\widetilde{S}$ is contained in some fibre $\varphi^{-1}(x_0)$ of the map φ. Consequently the image $F(R) \subset N$ of the branch locus is contained in the fiber over x_0 of the map $N \to \mathbb{P}^{n-d} \times \varphi(S) \xrightarrow{p_2} \varphi(S)$. But such a fiber consists of finitely many subvarieties of dimension $n - d$ and hence is of codimension $d \geq 2$. Let $\widetilde{U}$ be the complement of $F(R) \cup A$ in N and let $U \subset S$ be the Zariski open subset over which F is biregular. By the Lefschetz hyperplane section theorem [19] we can find a smooth projective curve M' contained in $\widetilde{U}$ whose fundamental group maps onto the fundamental group of $\widetilde{U}$. Since the complement U is a proper projective subvariety by standard arguments of topological transversality, the inclusion map induces an epimorphism $\pi_1(U) \to \pi_1(S) \to 1$. Combined with the fact that $F^{-1}: \widetilde{U} \to U$ is an isomorphism this yields that the fundamental group of $M := F(M')$ maps onto $\pi_1(S)$, and we can compose this epimorphism with ϱ to obtain a Zariski dense representation $\varrho': \pi_1(M) \to G$. The Higgs bundle (E_1, θ_1) corresponding to ϱ' is just the restriction $(E, \theta)|_M$. Taking the genericity of M' into account we may assume that (E_1, θ_1) is regular and semisimple. The induced spectral covering $\widetilde{M} = \widetilde{S}|_M \to M$ will be nonramified by the construction of M. On the other hand, every spectral covering over a curve has a branch locus which supports a divisor in some multiple of the canonical class of the curve. Thus $\widetilde{M} \to M$ can be nonramified only when it is totally disconnected covering. For a regular and semisimple Higgs bundle (E_1, θ_1)

this is equivalent to the fact that (E_1, θ_1) decomposes as a direct product of $\mathbb{C}^*$ bundles. But then the image $\varrho'(\pi - 1(M))$ is contained in some algebraic subtorus in G, which contradicts the Zariski density of ϱ'. q.e.d.

Now we are ready to obtain a map to an orbicurve. Consider the Stein factorization $S \to Y \to \varphi(S)$ of the map φ. The variety Y is a normal projective variety and has dimension one by Lemma 4.2. Thus $f: S \to Y$ is a map to a compact Riemann surface with connected fibers. Let $Q \subset Y$ be the finite set consisting of all points in which the map f is not smooth and of the fiber over x_0 of the covering $Y \to \varphi(S)$. Let $Y_0 = Y \backslash Q$ and $K = \varphi^{-1}(Y_0)$. Let $K_y = \varphi^{-1}(y)$ for some $y \in Y_0$. The space Y_0 is homotopic to a bouquet of circles (being an open Riemann surface). Thus $\pi_n(Y_0) = 0$ for $n \geq 2$, and from the long homotopy sequence of the fibration $f: K \to Y_0$ we get the following exact sequence:

$$1 \to \pi_1(K_y) \to \pi_1(K) \to \pi_1(Y_0) \to 1.$$

Furthermore the representation $\pi_1(K) \to \pi_1(S) \xrightarrow{\varrho} G$ which we will denote again by ϱ is trivial on $\pi_1(K_y)$. Indeed, denote by $\mathfrak{G}$ the Zariski closure of $\varrho(\pi_1(K_y))$ in G. The restriction $\varrho|_{\pi_1(K_y)}$ defines a regular and semisimple Higgs bundle (E_y, θ_y) over the smooth projective variety K_y. The associated spectral cover $p: \widetilde{K}_y \to K_y$ is the restriction of the covering $\widetilde{S}$ and hence is unramified. The pullback $p^*(E_y, \theta_y)$ is a Higgs bundle on $\widetilde{K}_y$ which decomposes as a product of $\mathbb{C}^*$-bundles. This can be seen as follows. Choose a regular and faithful representation ρ of G. Since (E_y, θ_y) is regular and semisimple, we can define an eigenvector line subbundle $\mathscr{L} \subset p^*\mathbf{V}$ of $p^*(\rho(\theta_y))$. The orbit of this bundle under the action of the Weyl group on $\widetilde{K}_y$ consists of $\#W$ different line bundles whose direct sum is naturally isomorphism to $p^*\mathbf{V}$. Going back to the associated principal bundles we obtain the desired decomposition. In this way the image $\varrho(\pi_1(\widetilde{K}_y))$ is contained in some torus T in G. Since the image of $\pi_1(\widetilde{K}_y)$ in $\pi_1(K_y)$ is a normal subgroup, this implies that $\mathfrak{G}$ is contained in the normalizer $n(T)$ of this torus. The semisimplicity of G guarantees that $n(T)$ is a proper subgroup, and therefore $\mathfrak{G}$ is also proper. But the exact sequence above shows that $\pi_1(K_y)$ is a normal subgroup in $\pi_1(K)$ which yields that $\mathfrak{G}$ is normal in G. Therefore $\mathfrak{G}$ should be trivial due to the fact that G is simple.

The orbifold structure of Y in the points of $Y \backslash Y_0$ is reconstructed by the same arguments as in [23]. This finishes the proof of the "if" part of the theorem.

2. The "only if" part follows easily from the standard properties of the spectral coverings of curves. Assume that (E, θ) is a pullback by the map $f: S \to Y$ of some Higgs bundle over the curve Y. Then the spectral covering $\widetilde{S}$ is the fibered product $\widetilde{S} = \widetilde{S} \times_f \widetilde{Y}$ of S with the corresponding Galois spectral covering of Y. Therefore the vector space of all 1-forms of type ε on $\widetilde{S}$ contains the vector space of all 1-forms of type ε on $\widetilde{Y}$. The latter can be identified with the space of all anti-invariant 1-forms on the double covering $\widetilde{Y}/W_{\text{even}} \to Y$, where W_{even} is the subgroup in W consisting of all elements with even length. By Hurwitz formula one can easily see that the space of the anti-invariant 1-forms of double covering of curves is always with positive dimension unless we are in the case of a nonramified covering of elliptic curves or in the case of covering of $\mathbb{P}^1$ ramified over two points.

The first case cannot occur since the covering $\widetilde{Y} \to Y$ is always ramified and the covering $\widetilde{Y}/W_{\text{even}} \to Y$ has the same branch locus. If the second case occurs again by the Hurwitz formula, it follows that $\widetilde{Y} = \widetilde{Y}/W_{\text{even}}$, i.e., $W = \mathbb{Z}_2$. The representation ϱ factors by hypothesis through a representation of the fundamental group of the orbicurve $(\mathbb{P}^1, \{x_1, x_2\}, \{2, 2\})$ which is isomorphic to $\mathbb{Z}_2$. This however contradicts the Zariski denseness of ϱ because there are no Zariski dense representations of finite groups.

Remark 4.1. Most of the above arguments were discovered by Simpson in his proof for the case $G = SL(2, \mathbb{C})$ from where we borrowed them. It is worthwhile to mention that the nontriviality of $\mathrm{Prym}_\varepsilon(\widetilde{S}, S)$ is automatic when $G = SL(2, \mathbb{C})$ because in this case the nontrivial 1-form ω on $\widetilde{S}$ is of type ε (ω is nontrivial because (E, θ) is regular and semisimple, which holds when ρ is nonrigid).

The characterization given in Theorem 4.1 of the representations factorizable through some curve is quite ineffective. To make the criterion more reliable we will describe a simple necessary and sufficient condition for the nontriviality of $\mathrm{Prym}_\varepsilon(\widetilde{S}, S)$.

As we saw in §3.3 the spectral covering $\widetilde{S}$ has a birational model $i_\theta(\widetilde{S})$ which is a subvariety in the total space $\mathrm{Tot}(\Omega^1_S)$. Actually from the construction of this model it is a subvariety in the total space of the subsheaf $i_\theta(t_E^\vee) \subset \Omega^1_S$. The following theorem shows that the subsheaf $i_\theta(\mathrm{t}_E^\vee)$ is an important invariant of the representation ϱ.

Theorem 4.2. *If (E, θ) is a regular and semisimple Higgs bundle associated with a Zariski dense representation of $\pi_1(S)$, then a necessary and sufficient condition for the nontriviality of* $\mathrm{Prym}_\varepsilon(\widetilde{S}, S)$ *is that the rank of the coherent sheaf $i_\theta(\mathrm{t}_E^\vee)$ must be equal to* 1.

Proof. The necessity is a straightforward consequence from Theorem 4.1. Indeed in this case the Higgs bundle (E, θ) will be a pullback of a Higgs bundle over some curve Y, and thus the section $\theta \in H^0(S, \mathfrak{g}_E \otimes \Omega^1_S)$ is actually a section in the sheaf $\mathfrak{g}_E \otimes f^* K_Y$. But then $i_\theta(\mathfrak{t}_E^\vee)$ is a subsheaf in the line bundle $f^* K_Y$ which was required to be shown.

To prove the sufficiency we have to show that there exists a nonzero 1-form of type ε on $\widetilde{S}$. One natural candidate for such a form is the form ω defined at the beginning of the proof of Theorem 4.1. Unfortunately the ε-part of ω is zero practically always. Indeed, we can give an alternative description of ω as the tautological section in the rank 1 subsheaf $L = p^*(i_\theta(\mathfrak{t}_E^\vee)) \subset \Omega^1_{\widetilde{S}}$. Let $\mathfrak{t}$ be the pullback of the sheaf $\mathfrak{t}_E^\vee$ on the total space $\mathrm{Tot}(\mathfrak{t}_E^\vee)$ and let ξ be the tautological section in $\mathfrak{t}$. One has the following commutative diagram

$$\begin{array}{ccc} \mathfrak{t} & \xrightarrow{i_{p^*(\theta)}} & L \\ & \searrow \quad \swarrow & \\ & \widetilde{S} & \end{array}$$

where the map $i_{p^*(\theta)}$ is a morphism of W-invariant sheaves over $\widetilde{S}$. Consequently the induced map $H^0(\widetilde{S}, \mathfrak{t}) \to H^0(\widetilde{S}, L)$ between the sections is a morphism of W-modules. Therefore the obvious relation $i_{p^*(\theta)}(\xi) = \omega$ yields by Schur's lemma that $i_{p^*(\theta)}$ is an isomorphism between the irreducible W modules generated by ξ and ω respectively. By the construction of $\mathfrak{t}$ it is clear that the module generated by ξ is isomoprhic to the module $\mathfrak{t}^\vee$ (here $\mathfrak{t}$ is an arbitrary Cartan subalgebra of $\mathfrak{g}$). This implies that the projection $P_\varepsilon(\omega)$ is not zero if and only if $\mathfrak{t} \simeq \varepsilon$ as W-modules, which occurs if and only if $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$.

Although the W-orbit of the form ω does not contain form whose ε part is not zero, we can make use of the subspace $\mathrm{Span}_{\mathbb{C}}(\mathrm{Orb}_W(\omega)) \subset H^0(\widetilde{S}, \Omega^1_{\widetilde{S}})$. Since $\dim \mathrm{Span}_{\mathbb{C}}(\mathrm{Orb}_W(\omega)) = \mathrm{rank}(\mathfrak{g}) > 1$, we can find a holomorphic 1-form $\omega' \in \mathrm{Span}_{\mathbb{C}}(\mathrm{Orb}_W(\omega))$ linearly independent of ω. Both forms ω and ω' are sections in the rank 1 subsheaf $L \subset \Omega^1_{\widetilde{S}}$ and therefore satisfy the condition $\omega \wedge \omega' = 0$. By the theorem of Castelnuovo-de Franchis (see, e.g., [5]) there exists a map with connected fibers from $\widetilde{S}$ to some curve $\widetilde{C}$ so that the forms ω and ω' are pullbacks from $\widetilde{C}$. Moreover one can identify the field of functions $\mathbb{C}(\widetilde{C})$ with the set of all meromorphic functions $\alpha \in \mathbb{C}(\widetilde{S})$ whose exterior derivative $d\alpha$ is a meromorphic section in L (cf. [5]). The coherent sheaf L is W-invariant by definition, and hence W acts on the curve $\widetilde{C}$ and the map $\widetilde{S} \to \widetilde{C}$ is W invariant. Set $C = \widetilde{C}/W$. Then the spectral covering $\widetilde{S}$ is a pullback of the covering of curves $\widetilde{C} \to C$. Since there is no element in W leaving the form ω fixed, it follows that the covering $\widetilde{C} \to C$ is Galois with Galois group W.

Again we would like to use the abundance of 1-forms of type ε for W-coverings of curves. As in the proof of the second part of Theorem 4.1 we can discard the two exceptional cases by observing that the covering $\widetilde{C} \to C$ is always ramified and that $C = \mathbb{P}^1$ may occur only when $G = SL(2, \mathbb{C})$. Consequently there is a nontrivial 1-form of type ε which we can pull back on $\widetilde{S}$. This finishes the proof of the theorem.

Theorem 4.2 indicated the importance of the rank of the sheaf $i_\theta(\mathfrak{t}_E^\vee)$ for the factorization question. The significance of this number is clarified a little bit further by the following statement.

Theorem 4.3. *Let ϱ be a Zariski dense representation of $\pi_1(S)$ whose Higgs bundle (E, θ) is regular and semisimple. Let* $\mathrm{rk}(i_\theta(\mathfrak{t}_E^\vee)) = i$. *If $i < \mathrm{rk}(\mathfrak{g})$, then there exist a normal projective variety C of dimension i and a holomorphic map $f\colon S \to C$ so that for certain Zariski open sets $U \subset S$ and $C_0 \subset C$ the induced representation $\varrho\colon \pi - 1(U) \to G$ factors geometrically through $\pi_1(C_0)$ via f.*

Proof. Let ω be again the 1-form defined at the beginning of the proof of Theorem 4.1. As we saw, the linear span of the W-orbit of ω is isomorphic to the Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$. By construction the W-orbit of the tautological section ξ of $\mathfrak{t}$ consists of sections which span the fiber of $\mathfrak{t}$ at the generic point. Since $L = i_{p^*(\theta)}(\mathfrak{t})$, this implies that the sections form $\mathrm{Orb}_W(\omega) \subset H^0(\widetilde{S}, L)$ will span the fiber of L at the generic point. By hypothesis $i = \mathrm{rk}\, L < \dim \mathfrak{t}$, and hence we can find a strict i-wedge subspace in $\mathrm{Span}_{\mathbb{C}}(\mathrm{Orb}_W(\omega))$. Using Catanese's generalization of the Castelnuovo-de Franchis theorem [5] with the same arguments as in Theorem 4.2 one can show that there exists a normal projective variety $\widetilde{C}$, with $W \subset \mathrm{Aut}(\widetilde{C})$ and a W-invariant map $\widetilde{S} \to \widetilde{C}$. Again the triviality of the stablizer $\mathrm{St}_W(\omega)$ yields that the fixed locus of each $w \in W$ is a proper subscheme in $\widetilde{C}$. Therefore the ramification divisor D for the covering $\widetilde{S} \to S$ maps onto a proper subscheme of $\widetilde{C}$.

Set $C = \widetilde{C}/W$ and consider the map $f\colon S \to C$. Let C_0 be the nonempty Zariski open subset in C obtained after throwing out the singular locus of C, the critical values of f, and the branch locus of the covering $\widetilde{C} \to C$. Set $U = f^{-1}(C_0)$. By Lefschetz's hyperplane section theorem and the long exact homotopy sequence of a fibration we get the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & 1 & & \\
 & & & & & & \downarrow & & \\
1 & \longrightarrow & \pi_1(F \cap H^{i-1}) & \longrightarrow & \pi_1(U \cap H^{i-1}) & \longrightarrow & \pi_1(C_0 \cap H^{i-1}) & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \downarrow \\
 & & & & \pi_1(F) & \longrightarrow & \pi_1(U) & \longrightarrow & \pi_1(C_0) \longrightarrow 1 \\
 & & & & \downarrow & & \downarrow & & \downarrow \\
 & & & & 1 & & 1 & & 1
\end{array}$$

where F is a fiber of $f\colon U \to C_0$, and $U \cap H^{i-1}$ denotes taking $i-1$ sufficiently general hyperplane sections in U. If we assume that the restriction of ϱ on $\pi_1(F)$ is nontrivial, then from the above diagram it will follow that $\varrho\colon \pi_1(F) \to G$ is Zariski dense, since as in the proof of Theorem 4.1 the normality of $\pi_1(F \cap H^{i-1})$ in $\pi_1(U \cap H^{i-1})$ will imply the Zariski denseness of $\varrho|_{\pi_1(F \cap H^{i-1})}$. Now the same argument as in the proof of Theorem 4.1 leads to a contradiction. Thus ϱ is trivial on $\pi_1(F)$, which finishes the proof of the theorem.

Remark 4.2. At the beginning of this section we required (E, θ) to be everywhere regular. This restriction on (E, θ) can be dropped without loss of generality. Indeed, using the Lefschetz hyperplane section theorem, by intersecting S by a general hyperplane H we get an isomorphism of the fundamental groups $\pi_1(S) \simeq \pi_1(S \cap H^i)$ for every $i \leq \dim S - 2$. Therefore, for the problem of factorizing representations geometrically we can restrict ourselves to the case where S is an algebraic surface. Since the set of nonregular elements in G has codimension at least 3, a standard count of parameters shows that over an algebraic surface the general Higgs field regular over a Zariski open set is regular everywhere.

Remark 4.3. After this work was finished the authors received the preprint [26] in which a similar problem is investigated in the case of $SL(3, \mathbb{C})$.

Remark 4.4. All of the proofs in this section can be modified for representations in semisimple Lie groups or for Higgs bundles which are only regular. We will omit the discussion of this generalizations since they differ only technically.

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